Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

Section 1: Hashing, Streaming

1 Open Addressing

Recall that in hashing with chaining, each table cell stores a bucket of keys. In *open addressing*, all keys are stored directly inside the table. When a collision occurs, we try another location according to some probe sequence.

Suppose we have a table of size $m = 11$, hash function

$$h(x) = x \bmod 11,$$

and linear probing rule

$$h_t(x) = (h(x) + t) \bmod 11$$

for $t = 0, 1, 2, \ldots$.

This means that the key x first tries slot $h_0(x) = h(x)$. If that slot does not answer the question, the algorithm tries $h_1(x)$, then $h_2(x)$, and so on. The sequence

$$h_0(x), h_1(x), h_2(x), \ldots$$

is called the *probe sequence* for x.

For this problem, assume there are no deletions. The operations work as follows:

- `put(x)` follows the probe sequence for x until it finds an empty slot, then stores x there. If it sees x along the way, it can stop because x is already present.
- `find(x)` follows the probe sequence for x until either it finds x, in which case it returns `True`, or it reaches an empty slot, in which case it returns `False`. Reaching an empty slot proves x is not present, because `put(x)` would have inserted x there before probing any later slots.

The current table is:

Index	0	1	2	3	4	5	6	7	8	9	10
T		12	23		15	5		18	29		21

(a) Draw the probe sequence used to insert 34, and show the updated table.

(b) Starting from the updated table, draw the probe sequence used to search for 56. Does the search succeed or fail?

(c) The table above was built using linear probing, so nearby collisions can form clusters. To analyze the probabilistic behavior more cleanly, suppose instead that each new key has a uniformly random probe sequence over the table locations. If the load factor is $\alpha = n/m$, what is the probability that the first probe collides? Under the simplifying assumption that each probe is an independent uniformly random location, what is the expected number of probes needed to insert a new key?

(d) Explain why open addressing becomes slow as α approaches 1, and briefly compare this with chaining.

2 [Bonus] Perfect Hashing

You are given a static set S of n keys from a universe $[U]$. Your goal is to preprocess S into a dictionary data structure that supports membership queries.

Design a randomized hashing-based data structure with the following guarantees:

- $O(n)$ expected preprocessing time,
- $O(n)$ space,
- $O(1)$ worst-case query time after preprocessing.

Your answer should describe the data structure, the preprocessing algorithm, the query algorithm, and why the stated time and space guarantees hold.

3 Streaming Algebra

You are given a continuous data stream of N integers, $A = (a_1, a_2, \ldots, a_N)$. You do not know N in advance, but you are guaranteed that $N \geq 4$. You may read the stream exactly once, and you may not store the array.

Your goal is to compute the maximum possible value of

$$a_i - 2a_j + 3a_k - 4a_l$$

over all choices of indices satisfying

$$1 \leq i < j < k < l \leq N.$$

Assume you are restricted to $O(\log S)$ bits of memory, where S is the maximum possible absolute value of the final expression.

Design a deterministic streaming algorithm for this problem. Your answer should state the variables your algorithm maintains, how those variables are updated when a new stream element arrives, why the algorithm is correct, and why it uses only $O(\log S)$ bits of memory.

Section 2: Graphs, Greedy Algorithms

4 Remove Covered Intervals

Given an array of intervals, remove all intervals that are covered by another interval in the list.

The interval $[a, b]$ is covered by the interval $[c, d]$ if and only if $c \leq a$ and $b \leq d$.

Find the number of remaining intervals.

Example: $[[1, 5], [5, 20], [3, 12], [4, 12]]$ In this example, the interval $[3, 12]$ fully covers the interval $[4, 12]$. Hence, the remaining intervals would be $[[1, 5], [5, 20], [3, 12]]$.

5 Money Madness

Anirudh is on vacation but forgot to convert his US dollars (USD) to Indian Rupees (INR)! He visits a currency exchange booth at the airport. The booth provides an exchange rate to convert between various different currencies. Unfortunately, the exchange rate provided is never better than the actual conversion rate (you will never make money by exchanging).

(a) Given a list of exchange rates, provide an algorithm to find the maximum amount of INR he can get for 1 USD.

(b) Now, assume the booth charges a percentage fee (varies depending on currency) for each exchange. For example, if 1 USD is worth 100 INR, the currency exchange might only give you 90 INR per dollar. We represent this as an exchange rate of 0.9, because you retain 90% of the value. Once again, all exchange rates are in the range $(0, 1)$ exclusive (so you cannot profit).

Design an algorithm that finds the best sequence of exchanges for converting USD to INR (using any number of exchanges), retaining as much value as possible.

Section 3: Linear Programming and Zero-Sum Games

6 Review Planning

You have T hours left to study for an oral exam. There are m review activities you can choose from, such as redoing a homework problem, reading lecture notes, going through a discussion worksheet, or explaining a proof to a friend.

There are k topics on the exam. Spending one hour on activity j gives a_{ij} units of preparation for topic i. You may spend a fractional number of hours on each activity. Let $x_j \geq 0$ be the number of hours that you spend on activity j.

You expect the examiner to look for areas where you are least prepared, so you want to make your study plan balanced: your goal is to maximize your weakest topic preparation, which is given by

$$\min_{i \in [k]} \sum_{j=1}^{m} a_{ij} x_j.$$

(a) Formulate this study planning problem as a linear program.

(b) Write down the dual of your LP from part (a). Briefly describe an interpretation of the dual from the examiner's perspective.

7 Penalty Kicks

A kicker and a goalie are playing a zero-sum game. The kicker chooses where to shoot, the goalie chooses where to dive, and the payoff is the probability that the kicker scores. The kicker wants to maximize this probability, while the goalie wants to minimize it.

The payoff matrix is

	Dive Left	Stay Center	Dive Right
Shoot Left	0.2	0.7	0.8
Shoot Center	0.4	0.5	0.4
Shoot Right	0.8	0.7	0.2

(a) Eliminate any dominated strategies in order to reduce this 3×3 game to a 2×2 game. Justify each strategy you eliminate.

Hint: One strategy may be dominated by a mixture of other strategies.

(b) Solve the resulting 2×2 zero-sum game. What is the kicker's optimal mixed strategy, what is the goalie's optimal mixed strategy, and what is the value of the game?

(c) Write a linear program for the kicker's optimal mixed strategy in the original 3×3 game, and give the optimal solution to this linear program.

(d) Suppose a different kicker is better at shooting left, and has the payoff matrix

	Dive Left	Stay Center	Dive Right
Shoot Left	0.3	0.8	0.9
Shoot Center	0.4	0.5	0.4
Shoot Right	0.8	0.7	0.2

Would the solution to the zero-sum game still involve a mixed strategy, or should the kicker now always shoot left? Explain in one or two sentences.

Section 4: NP-Completeness

8 Bad Reductions

In each part we make a wrong claim about some reduction. Explain for each one why the claim is wrong.

(a) The shortest simple path problem with non-negative edge weights can be reduced to the longest simple path problem by just negating the weights of all edges. There is an efficient algorithm for the shortest simple path problem with non-negative edge weights, so there is also an efficient algorithm for the longest path problem.

(b) Undirected Rudrata Path can be reduced to Longest Path in a DAG in the following way: Given the undirected graph G, we will use DFS to find a traversal of G and assign directions to all the edges in G based on this traversal. In other words, the edges will point in the same direction they were traversed and back edges will be omitted, giving us a DAG. If the longest path in this DAG has $|V| - 1$ edges, then there must be a Rudrata path in G, as any simple path with $|V| - 1$ edges must visit every vertex.

9 Exact 4-SAT

The Exact 4-SAT problem is defined as follows.

Input: n boolean variables $\{x_1, \ldots, x_n\}$ and clauses $\{C_1, \ldots, C_m\}$ with each containing exactly {four distinct} literals. For example, the following is an instance of Exact 4-SAT,

$$(x_1 \vee x_2 \vee \overline{x_4} \vee \overline{x_5}) \wedge (x_3 \vee \overline{x_4} \vee \overline{x_1} \vee x_2) \wedge (x_1 \vee x_3 \vee x_4 \vee x_5)$$

Note that all the 4 literals within a clause have to be distinct.

Goal: Find an assignment to the variables $x_1, \ldots, x_n$ that satisfies all the clauses.

(a) Give a polynomial time reduction from 3-SAT to Exact 4-SAT.

(b) Give a polynomial time reduction from Exact 4-SAT to 3-SAT.

(c) What does this tell us about the complexity of Exact 4-SAT?

10 Dominating Set

A dominating set of a graph $G = (V, E)$ is a subset S of V, such that every vertex not in S is a neighbor of at least one vertex in S. Let the Dominating Set problem be the task of determining whether there is a dominating set of size $\leq k$. We will show that the Dominating Set problem is NP-Complete. You may assume that G is connected.

(a) Show that Dominating Set is in NP.

(b) Show that Dominating Set is NP-Hard.
Hint: Try reducing from Vertex Cover or Set Cover.

11 Independent Set Approximation

In the Max Independent Set problem, we are given a graph $G = (V, E)$ and asked to find the largest set $V' \subseteq V$ such that no two vertices in V' share an edge in E.

Given an undirected graph $G = (V, E)$ in which each node has degree $\leq d$, give an efficient algorithm that finds an independent set whose size is at least $1/(d + 1)$ times that of the largest independent set. Describe your algorithm and prove that the it finds an independent set of size is at least $1/(d+1)$ times the largest possible solution. Your algorithm should run in time $O(|V| \cdot |E|)$ (or less).

Section 5: Divide & Conquer, FFT, Parallelism, Dynamic Programming

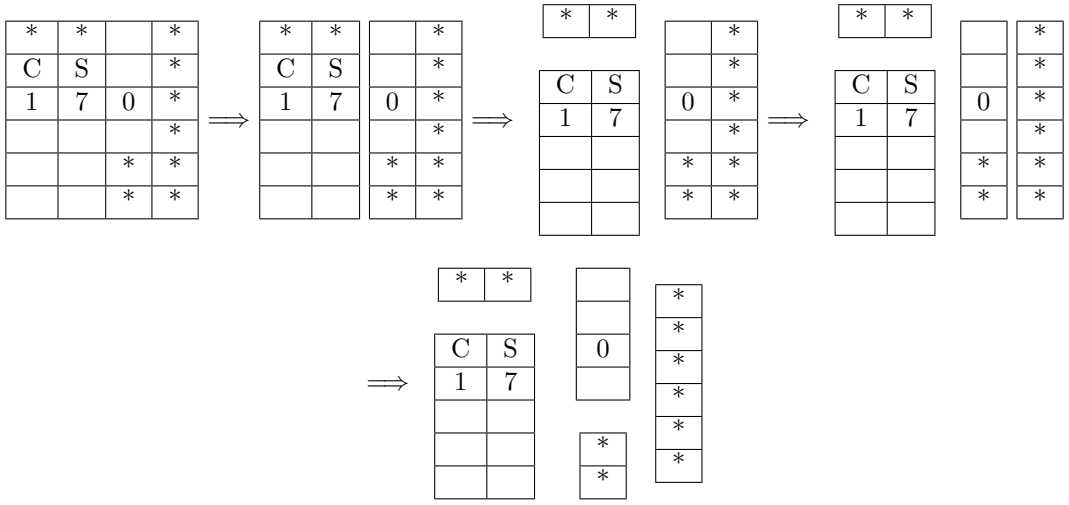
12 Sorted Array

Given a sorted array A of n (possibly negative) distinct integers, you want to find out whether there is an index i for which $A[i] = i$. Devise a divide-and-conquer algorithm that runs in $O(\log n)$ time.

13 My Dog Ate My Homework

One morning, you wake up to realize that your dog ate some of your CS 170 homework paper, which is an $\ell \times w$ rectangular grid of squares. Some of the squares have holes chewed through them, and you cannot use paper that has a hole in it. You would like to cut the paper into pieces so as to separate all the tattered squares from all the clean, un-bitten squares. You want to do this so that you can save as much of your work as possible.

For example, shown below is a 6×4 piece of paper where the bitten squares are marked with *. As shown in the picture, one can separate the bitten parts out in exactly four cuts.



(Each *cut* is either horizontal or vertical, and of one piece of paper at a time.)

Formally, the problem is as follows:

> **Input:** Dimensions of the paper $\ell \times w$ and an array $A[i, j]$ such that $A[i, j] = 1$ if and only if the ij^{th} square has holes bitten into it.
>
> **Goal:** Find the minimum number of cuts needed so that the $A[i, j]$ values of each piece are either all 0 or all 1.

Design a DP-based algorithm to find the smallest number of cuts needed to separate all the bitten parts out in $O(\ell^3 w^3)$ time.

